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Volterra Integral Equations and Some Nonlinear Integral Equations with Variable Limit of Integration as Generalized Moment Problems

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Abstract: In this paper we will see that, under certain conditions, the techniques of generalized moment problem will apply to numerically solve an Volterra integral equation of first kind or second kind. Volterra integral equation is transformed into a one-dimensional generalized moment problem, and shall apply the moment problem techniques to find a numerical approximation of the solution. Specifically you will see that solving the Volterra integral equation of first kind $f(t)=\int_a^t K(t,s)x(s)ds\ a\le t\le b$ or solve the Volterra integral equation of the second kind $x(t)=f(t)+\int_a^t K(t,s)x(s)ds\ a\le t\le b$ is equivalent to solving a generalized moment problem of the form $\mu_n=\int_a^b g_n(s)x(s)ds\ n=0,1,2,\ldots.$ This shall apply for to find the solution of an integrodifferential equation of the form $x'(t)=f(t)+\int_a^t K(t,s)x(s)ds$ for $a\le t\le b$ and $x(a)=a_0$ Also considering the nonlinear integral equation: $f(x)=\int_a^x y(x-t)y(t)dt$ This integral equation is transformed a two-dimensional generalized moment problem. In all cases, we will find an approximated solution and bounds for the error of the estimated solution using the techniques ofgeneralized moment problem.

Key words: Generalized moment problems, solution stability, Volterra integral equations, nonlinear integral equations.

1. Introduction

An equation of the form

$$\boldsymbol{x(t)=f(t)+\lambda\int_a^t K(t,s)x(s)ds\ \ a\le t\le b}$$

where $\boldsymbol{f(t)}$ *y* $\boldsymbol{K(t,s)}$ are known functions, $\boldsymbol{\lambda}$ is a numerical parameter and $\boldsymbol{x(t)}$ is a unknown function, is a Volterra integral equation of second kind. The function $\boldsymbol{K(t,s)}$ is the kernel of the Volterra integral equation. If $\boldsymbol{f(t)=0}$ then the integral equation is said homogeneous.

The equation

$$\boldsymbol{f(t)=\int_a^t K(t,s)x(s)ds\ \ a\le t\le b}$$

where $\boldsymbol{x(t)}$ is the unknown function, is a Volterra integral equation of first kind. In many scientific and engineering problems of Volterra integral equations are present and have attracted much attention to find analytical and numerical methods for their solution. Some applications of Volterra integral equations are: population dynamics, spread of epidemics, semiconductor devices, inverse problems, etc.

One of the fundamental methods of solving Volterra integral equations of second kind is the method of resolvents [1], [9] where the solution is given by

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$$x(t) = f(t) + \lambda \int_a^t R(t,s,\lambda) f(s) ds$$

The $R(t,s,\lambda)$ is the resolvent function of the integral equation and is defined as the sum of the series

$$R(t,s,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(t,s)$$

wherein the cores iterated $K_{n+1}(t,s)$ satisfy a recurrence relation.

Another fundamental method it is the method of successive approximations [1], where the solution is determined as the limit of the sequence $\{x_n(t)\}_n$ $n = 0,1,2,\ldots$ whose general term is found by the recurrence formula

$$x_n(t) = f(t) + \lambda \int_a^t K(t,s) x_{n-1}(s) ds$$

Other methods of resolution [10], [11], [12], involving the Laplace transform, are used to solve Volterra integral equations of convolution

$$x(t) = f(t) + \lambda \int_a^t K(t-s) x(s) ds$$

Volterra integral equations of first kind, under certain conditions, can be reduced to a Volterra integral equation of second kind.

2. The Generalized Moment Problem

The generalized moment problem [2], [6], [7], [8] is defined as finding the function $\boldsymbol{f(x)}$ on a domain $\boldsymbol{\Omega} \subset \boldsymbol{R^d}$ satisfying the equations

$$\boldsymbol{\mu_n = \int_\Omega g_n(x) f(x) dx \ n \in N} \qquad (1)$$

where $(\boldsymbol{g_n})$ is a given sequence of functions in $\boldsymbol{L^2(\Omega)}$ linearly independent. The moment problem is an ill-conditioned problem. There are several methods for constructing regularized solutions. One is the method of truncated expansion.

The truncated expansion method consists in approximating (1) by finite moment problem

$$\mu_i = \int_\Omega g_i(x) f(x) dx \ i = 1,2,\ldots,n \qquad (2)$$

In the subspace generated by $g_1, g_2, \ldots, g_n$ the solution is stable. In the case where the data $\mu_1, \mu_2, \ldots, \mu_n$ are inexact convergence theorems and error estimates for the regularized solutions must be applied.

It can be proved that [8] a necessary and sufficient condition for the existence of a solution of (1) is that

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} C_{ij}\, \mu_j \right)^2 < \infty$$

where C_{ij} are given by (13).

3. Volterra Linear Integral Equation of Second Kind

We want to find a function $\boldsymbol{x(t)} \in \boldsymbol{L^2(a,b)}$ such that

$$\boldsymbol{x(t) = f(t) + \int_a^t K(t,s) x(s) ds \ \ a \le t \le b} \qquad (3)$$

where

$\boldsymbol{f(t)} \in \boldsymbol{L^2(a,b)}$ and $\boldsymbol{K(t,s)} \in \boldsymbol{L^2(R)}$

$\boldsymbol{R} = (\boldsymbol{a},\boldsymbol{b}) \times (\boldsymbol{a},\boldsymbol{b})$ are known functions.

Theorem 1: *If* $\boldsymbol{f(t)} \in \boldsymbol{L^2(a,b)}$ *and* $\boldsymbol{K(t,s)} \in \boldsymbol{L^2(R)}$ $\boldsymbol{R} = (\boldsymbol{a},\boldsymbol{b}) \times (\boldsymbol{a},\boldsymbol{b})$ *then* (3) *has a unique solution in* $\boldsymbol{L^2(a,b)}$ [1].

To write (3) as a moment problems:

$$\boldsymbol{-f(t) = -x(t) + \int_a^t K(t,s) x(s) ds} \qquad (4)$$

We take a basis $\{\boldsymbol{\psi_n(t)}\}_{\boldsymbol{n}}$ in $\boldsymbol{L^2(a,b)}$ and both sides of (4) are multiplied by $\boldsymbol{\psi_n(t)}$ and integrated between *a* and *b*

$$\boldsymbol{\int_a^b -f(t)\psi_n(t)\, dt}$$

$$\boldsymbol{= \int_a^b -x(t)\psi_n(t) dt}$$

$$\boldsymbol{+ \int_a^b \int_a^t K(t,s) x(s) \psi_n(t)\, ds\, dt}$$

We call $\boldsymbol{\int_a^b -f(t)\psi_n(t)\, dt = \mu_n}$

In addition

$$\int_a^b \int_a^t \boldsymbol{K(t,s)x(s)\psi_n(t)\,ds\,dt}$$

$$= \int_a^b \boldsymbol{x(s)} \int_s^b \boldsymbol{K(t,s)\psi_n(t)\,dt\,ds}$$

$$= \int_a^b \boldsymbol{x(s)g_n^*(s)ds}$$

Hence

$$\mu_n = \int_a^b -x(t)\psi_n(t)dt + \int_a^b x(s)g_n^*(s)ds$$

$$= \int_a^b -x(t)\psi_n(t)\,dt$$

$$+ \int_a^b x(t)g_n^*(t)dt$$

$$= \int_a^b x(t)[-\psi_n(t) + g_n^*(t)]\,dt$$

$$= \int_a^b x(t)G_n^*(t)dt$$

Consequently

$$\boldsymbol{\mu_n = \int_a^b x(t)G_n^*(t)dt\ n \epsilon N} \qquad (5)$$

If $\{\boldsymbol{G_n^*(t)}\}_n$ are linearly independent is solved (5) as a generalized moment problem.

Let us see under what conditions $\{G_n^*(t)\}_n$ are linearly independent.

We have $g_n^*(s) = \int_s^b K(t,s)\psi_n(t)\,dt$.

Further $-\psi_n(s) + g_n^*(s) = G_n^*(s)$

We consider the operator

$$L(\varphi) = -\varphi + \int_s^b K(t,s)\varphi(t)\,dt$$

Then $L(\varphi)$ is linear. If L is nonsingular, that is $L(\varphi) = 0 \implies \varphi = 0$, then L preserves the linear independence.

In this case $\{L(\psi_n)\}_n = \{G_n^*\}_n$ would be linearly independent.

But $L(\varphi) = 0$ can be viewed as a Volterra integral equation of second kind with $f(s) = 0$

$$\boldsymbol{0 = -\varphi(s) + \int_s^b K(t,s)\varphi(t)\,dt \implies}$$

$$\boldsymbol{\varphi(s) = \int_s^b K(t,s)\varphi(t)\,dt}$$

$$\boldsymbol{\varphi(s) = \int_b^s (-K(t,s))\varphi(t)\,dt} \qquad (6)$$

If we assume that

$\boldsymbol{K(t,s) \in L^2(R)\ R = (a,b) \times (a,b)}$

then as $\boldsymbol{\varphi(s) = 0}$ is solution of (6) by the previous theorem is the only solution of (6) in $\boldsymbol{L^2(a,b)}$.

Consequently L is nonsingular.

4. Volterra Linear Integral Equation of First Kind.

We want to find a function $\boldsymbol{x(t) \in L^2(a,b)}$ such that

$$\boldsymbol{f(t) = \int_a^t K(t,s)x(s)ds\ a \le t \le b} \qquad (7)$$

with $\boldsymbol{f(t) \in L^2(a,b)}$ and $\boldsymbol{K(t,s) \in L^2(R)\ R = (a,b) \times (a,b)}$ known functions.

The following result can be seen in [1]:

Given a Volterra integral equation of first kind, it can be written as a Volterra integral equation of second kind by applying derivation in (7) with respect to t

$$\boldsymbol{f'(t) = \int_a^t K_t(t,s)x(s)ds + K(t,t)x(t)} \qquad (8)$$

we write $\boldsymbol{K^*(t,s) = \frac{K_t(t,s)}{K(t,t)}}$ and $\boldsymbol{f^*(t) = \frac{f'(t)}{K(t,t)}}$ then

$$f^*(t) = \int_a^t K^*(t,s)x(s)ds + x(t) \qquad (9)$$

from here (9) is a Volterra integral equation of second kind.

In the case of $K(t,t) = 0$ then remains (8) an equation of first kind. It derives again until that $K_t^{(n)}(t,t) \neq 0$.

We must have $K(t,s)$ and $x(t)$ continuous functions in their respective domains, $K(t,s)$ and $f(t)$ differentiables functions with respect to *t*, and it must also be continuous $K_t(t,s)$.

Thus if $K(t,t) \neq 0$ on *(a,b)* and taking into account that it must be $f(a) = 0$, (7) is equivalent to (9).

If $K^*(t,s) \in L^2(R)$ and $f^*(t) \in L^2(a,b)$ then (9) (and (7)) has a unique solution in $L^2(a,b)$.

If we have

$$f(t) = \int_a^t K(t,s) g(x(s)) ds \quad a \leq t \leq b \qquad (10)$$

so with the above arguments we arrive at

$$f^*(t) = \int_a^t K^*(t,s) g(x(s)) ds + g(x(t)) \qquad (11)$$

In this case (11) is analogous to the generalized moment problem

$$\mu_n = \int_a^b g(x(t)) G_n^*(t) dt \; n \in N \qquad (12)$$

To solve numerically (5) as a generalized moments problem, truncated expansion method detailed in [3] and generalized in [5] is applied for the corresponding finite problem with $i = 0,1,2,\ldots,N$. We write $p_N(t)$ to approximate $x(t)$.

Is taken a base $\varphi_i(t)\; i = 0,1,2,\ldots$ of $L^2(a,b)$ obtained from the sequence $G_i^*(t)\; i = 0,1,2,\ldots,N$ by Gram-Schmidt method and necessary functions are added in order to have an orthonormal basis..

We then approximate the solution *x(t)* with [5]:

$$p_N(t) = \sum_{i=0}^{N} \lambda_i \, \varphi_i(t)$$

$$\text{where } \lambda_i = \sum_{j=0}^{i} C_{ij} \;\; \mu_j \;\; i = 0,1,\ldots,N$$

And the coefficients C_{ij} verifies

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle G_i^*(t) \mid \varphi_k(t) \rangle}{\|\varphi_k(t)\|^2} C_{kj} \right) \cdot \|\varphi_i(t)\|^{-1} \qquad 1 < i \leq N \quad ; \quad 1 \leq j < i$$

$$C_{ii} = \|\varphi_i(t)\|^{-1} \qquad i = 0,1,\ldots,N \qquad (13)$$

It can be seen in [5] the following theorem

Theorem 2: *Let* $\{\mu_k\}_{k=0}^{N}$ *be a set of real numbers and suppose that x(t)* $\in L^2(a,b)$ *verifies for some N* ε *and E (two positive numbers)* $\sum_{k=0}^{N} \left| \int_a^b G_k^*(t) x(t) dt - \mu_k \right|^2 \leq \varepsilon^2$ *and*

$\int_a^b |x'(t)|^2 dt \leq E^2$ *then*

$$\int_a^b |p_N(t) - x(t)|^2 dt \leq \|C^T C\| \varepsilon^2 + \frac{(b-a)^2}{4(N+1)^2} E^2$$

where C is the matrix with coefficients given by (13)

If we apply the truncated expansion method to solve the equation (11) would obtain an approximation $p_N(t)$ for $g(x(t))$.

Thus if g^{-1} is continuous, then $g^{-1}(p_N(t))$ is an estimate of $x(t)$

And if g^{-1} is Lipschitz in a domain D that includes the image of $x(t)$, ie if

$$\|g^{-1}(x) - g^{-1}(y)\| \leq \lambda \|x - y\|$$

for some λ and $\forall x, y \in D$ then

$$\int_a^b \left| g^{-1}(p_N(t)) - x(t) \right|^2 dx$$

$$\leq \lambda \left(\|C^T C\| \varepsilon^2 + \frac{(b-a)^2}{4(N+1)^2} E^2 \right)$$

5. Application

Suppose the integrodifferential equation

$$x'(t) = f(t) + \int_a^t K(t,s) x(s) ds \quad a \leq t \leq b \quad (14)$$

with initial condition $x(a) = a_0$.

We integrate from *a* to *t*

$$\int_a^t x'(t) dt = \int_a^t f(t) dt + \int_a^t \int_a^t K(t,s) x(s) ds \, dt$$

Thus

$$x(t)-x(a)=F(t)-F(a)+\int_a^t\int_s^t K(t,s)x(s)dtds$$

where $F(t)$ is the primitive of $f(t)$.

If we write $K^p(t,s)$ the primitive of $K(t,s)$ with respect to *t*, then

$$x(t)-x(a)=F(t)-F(a)+\int_a^t [K^p(t,s) - K^p(s,s)]x(s)ds$$

If we replace $F(t)-F(a)+x(a)=G(t)$ and $K^p(t,s)-K^p(s,s)=K^*(t,s)$ then

$$x(t)=G(t)+\int_a^t K^*(t,s)x(s)ds \qquad (15)$$

That is to say leads to a Volterra integral equation of second kind. Therefore resolver (14) is equivalent to solving (15).

6. Nonlinear Integral Equation with Variable Limit of Integration

Suppose we want to find $\boldsymbol{y(t)\ \epsilon\ L^2(a,b)}$ such that

$$\int_a^x \boldsymbol{y(t)y(x-t)dt=f(x)}\ \ \boldsymbol{a\le x\le b} \qquad (16)$$

with $\boldsymbol{f(x)\in L^2(a,b)}$ known.

We take a basis of $\boldsymbol{L^2(a,b)}$. We multiply both sides of (16) and integrate between *a* and *b*:

$$\boldsymbol{\mu_n}=\int_a^b \boldsymbol{f(x)\psi_n(x)dx}$$

$$=\int_a^b\int_a^x \boldsymbol{y(t)y(x-t)\psi_n(x)dtdx}$$

Then

$$\int_a^b\int_a^x \boldsymbol{y(t)y(x-t)\psi_n(x)dtdx}$$

$$=\int_a^b\int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_n(t+s)dsdt}$$

Consequently

$$\int_a^b\int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_n(t+s)dsdt=\mu_n\ n\epsilon N} \quad (17)$$

It can be considered (17) as a two-dimensional generalized moments problem over a region

$$\boldsymbol{\Omega=\{(t,s);\ a\le s\le b-t\,;\ a\le t\le b\}}$$

with $g_n(t,s)=\psi_n(t+s)$ and the unknown function is $x(t,s)=y(t)y(s)$

It is chosen $\{\psi_n(t)\}_n$ such that $\{\psi_n(t+s)\}_n$ are linearly independent.

By solving the corresponding finite problem

$$\int_a^b\int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_i(t+s)dsdt=\mu_i\ i=0,1,\dots,N}$$

applying the truncated expansion method we find the approximation $p_N(t,s)$ for $y(t)y(s)$.

Thus $p_N(t,t)$ will be an approximation of $y^2(t)$. Consequently $\sqrt{p_N(t,t)}$ will be an estimate of $y(t)$

Theorem 2 can be adapted to the case of a two-dimensional moments problem [4] considering a rectangular region *R* such that $\Omega\subset R$.

Note that if $f(x)\in L^2(a,b)$ then (17) , and therefore (16), has a solution in $L^2(a,b)$ because

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} C_{ij}\,\mu_j\right)^2=\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} C_{ij}\int_a^b f(x)\psi_j(x)\right)^2=$$

$$\sum_{i=1}^{\infty}\left(\int_a^b f(x)\sum_{j=1}^{i} C_{ij}\,\psi_j(x)\right)^2=\|f(x)\|^2<\infty$$

7. Numerical Examples

7.1 We Consider the Volterra Integral Equation of Second Kind

$$\boldsymbol{x(t)=1-t-\frac{3}{2}t^2+\frac{t^3}{3}+\int_0^t\left(\frac{1+t}{1+s}\right)x(s)ds\ \ 0}<t<1$$

The solution is $x(t)=1-t^2$. Was taken $N=6$ and

$$\psi_n(t)=t^n\ n=0,1,2,\dots,N$$

The approximate solution is

$$p_6(t) = \frac{1}{1+t}(0.9999147621585023 + 1.0049081485367966t - 1.0672475817558449t^2 - 0.6253280251875469t^3 - 1.0184797250168018t^4 + 1.4149009386296558t^5 - 0.9279894745774144t^6 + 0.17611185238095847t^7 + 0.04332685368242366t^8)$$

Theorem 2 provides an estimate of the "accuracy" of the approximate solution. Is calculated for the example given

$$\|p_6(t) - x(t)\| = 0.0000182523$$

7.2 We Consider the Volterra Integral Equation of First Kind

$$\boldsymbol{t^2 = \int_0^t e^{t+s}\, x(s) ds\ 0} < t < 1$$

The solution is $x(t) = e^{-2t}(2t - t^2)$.

Was taken $N = 6$ and

$$\psi_n(t) = t^n\ n = 0,1,2,\dots,N$$

The approximate solution is

$$p_6(t) = -83.95739831942318 + 83.95743097212348e^t - 81.95933679986479t - 46.95163623263498t^2 - 8.153902721602144t^3 - 7.676928797912827t^4 + 1.1473834615675322t^5 - 0.5328318171899457t^6$$

and $\|p_6(t) - x(t)\| = 7.74288 \times 10^{-6}$

7.3 We Consider the Integrodifferential Equation

$$\boldsymbol{x'(t) = 2t - \frac{t^5}{4} - t^3 + \int_0^t st\, x(s) ds\ 0} < t < 1$$

The solution is $\boldsymbol{(t) = t^2 + 2}$.

Was taken $N = 6$ and $\boldsymbol{\psi_n(t) = t^n}\ n = 0,1,2,\dots,N$

The approximate solution is

$$p_6(t) = 2.000011350675333 - 0.0006000805526966135t + 1.0074722441931296t^2 - 0.03672718024767407 4t^3 + 0.08179727671799264t^4 - 0.07332085167364494t^5 - 0.008280528273741879t^6 + 0.05157218926604704t^7 - 0.02138479219837439t^8 + 0.0003789028271436808t^9 - 0.000934671238187303t^{10}$$

and $\|p_6(t) - x(t)\| = 3.57206 \times 10^{-6}$

7.4 We Consider the Integral Equation

$$\frac{1}{2} sen(x) = \int_0^x y(t) y(x-t) dt\ 0 < x < 1$$

The solution is $\boldsymbol{y(x) = \sqrt{\frac{1}{2}}\, J_0(x)}$ where

$$J_0(x) = BesselJ(0, x)$$

Was taken $\boldsymbol{N = 4}$ and

$$\psi_n(t) = t^n\ n = 0,1,2,\dots,N$$

The approximate solution is

$$p_4(x) = \Bigg[0.4596976941318603 - 0.09533610866670236\left(-\frac{2}{3} + 2x\right) - 0.07464930125266528\left(-\frac{1}{2} + 4x^2 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right)\right) + 0.008845907552301498\left(-\frac{2}{5} + 8x^3 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right) - \frac{12}{7}\left(-\frac{1}{2} + 4x^2 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right)\right)\right)\Bigg]^{\frac{1}{2}}$$

and $\|\boldsymbol{p_4(x) - y(x)}\| = \mathbf{0.000959458}$

8. Conclusions

Given a Volterra integral equation of second kind of the form

$$\boldsymbol{x(t) = f(t) + \int_a^t K(t,s)x(s)ds \;\; a \le t \le b}$$

with $\boldsymbol{f(t) \in L^2(a,b), K(t,s) \in L^2(R)}$ $\boldsymbol{R = (a,b) \times (a,b)}$ known functions, it can be written as a one-dimensional generalized moments problem, and can apply the techniques of moments problem to find a numerical approximation of the solution.

The Volterra integral equation of first kind

$$\boldsymbol{f(t) = \int_a^t K(t,s)x(s)ds \;\; a \le t \le b}$$

where $f(t) \in L^2(a,b)$, $K(t,s) \in L^2(R)$ $R = (a,b) \times (a,b)$ known functions, can be written as a Volterra integral equation of second kind if $K(t,t) \neq 0$ on (a,b) , $f(a) = 0$, $K_t(t,s)$ and $f'(t)$ continuous functions in their respective domains.

The nonlinear integral equation

$$\int_a^x y(t)y(x-t)dt = f(x) \;\; a \le x \le b$$

with $f(x) \in L^2(a,b)$ known function, can be written as a two-dimensional generalized moments problem.

In all cases the moments problem techniques can be applied to find a numerical approximation of the solution.

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